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A GENERALISED CONWAY-NORTON RELATIONSHIP BETWEEN THE MONSTER AND CONWAY GROUPS

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ABSTRACT

We conjecture a general relationship between the centraliser of a conjugacy class of the Conway group and the centraliser of a corresponding Monster group conjugacy class for 51 classes. This generalises a well-known observation of Conway and Norton for 5 prime ordered classes. For each such class in the Monster, we also simply relate the Thompson series to the eta modular function of the corresponding Conway group class. A string theory interpretation for these relationships is briefly discussed.

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Introduction.

Conway and Norton in their famous Monstrous Moonshine paper [1] noted a relationship between the centraliser of 5 prime p ordered conjugacy classes of the Conway group [c.f. 2] (where $(p-1)|24$) and the centraliser of 5 conjugacy classes of the Fischer-Griess Monster group [3]. In this paper we propose a generalisation of this relationship valid for 51 Conway group classes which are characterised by two simple constraints. We begin with a brief discussion on the Conway group of Leech lattice automorphisms and their corresponding eta modular forms. The 51 classes of interest are then defined together with an associated set of 51 Monster group classes. The Monster group Thompson series [4] is then related, for each such class, to the corresponding Conway group eta modular form in a simple way. We next conjecture a generalised Conway-Norton relationship between the centraliser of each Conway group class and the centraliser of a Monster group class. We conclude with a number of remarks concerning this conjecture. This includes a brief description of a string theory interpretation for these observations where we suggest that the Moonshine module of Frenkel, Lepowsky and Meurman (FLM) [5,6] can be reproduced by a string theory orbifold construction [7,8] based on any of the 51 Conway group classes considered [9].

Leech lattice automorphisms.

Let Λ be the Leech lattice, the unique even self-dual lattice in 24 dimensions without elements of length squared two [c.f. 2]. Let $\overline{g}$ denote an element of the automorphism group of the lattice, the Conway group .0 i.e. $\overline{g}:\Lambda\to\Lambda$ where $<\overline{g}\alpha,\overline{g}\beta>=<\alpha,\beta>$ with a Euclidean inner product. Each conjugacy class of .0 can be distinguished by the characteristic equation for $\overline{g}$

$$\det\,(x-\overline{g})=\prod_{k|n}(x^k-1)^{g_k} \tag{1a}$$

$$\sum_{k|n}kg_k=24 \tag{1b}$$

where n is the order of $\overline{g}$ and the parameters $\{g_k\}$ are (not necessarily positive) integers. A complete list of all such characteristic equations is available in ref.[10]. Each conjugacy class is determined by the parameters $\{g_k\}$ and is usually presented in the Frame shape notation $a^{g_a}b^{g_b}.../c^{-g_c}d^{-g_d}...$ for $g_a, g_b, ... > 0$ and $g_c, g_d, ... < 0$ e.g. the reflection automorphism $\bar{r} : \alpha \to -\alpha$ with the characteristic equation $(x^2-1)^{24}/(x-1)^{24}$ and $g_2 = -g_1 = 24$ belongs to the class with Frame shape $2^{24}/1^{24}$.

We can associate with each conjugacy class the eta modular form

$$
\begin{aligned}
\eta_{\bar{g}}(\tau) =& \prod_{k|n}(\eta(k\tau))^{g_k} \\
=& q(1-g_1 q + O(q^2))
\end{aligned}
\tag{2}
$$

where $q = e^{2\pi i\tau}$, $\tau \in H$, the upper half complex plane and $\eta(\tau) = q^{1/24}\prod_n(1-q^n)$ is the Dedekind eta function [e.g. 11]. Using the modular properties of $\eta(\tau)$, it is useful to note that

$$
\begin{aligned}
\eta_{\bar{g}}(-1/\tau) =& D^{-1/2}(-i\tau)^{d/2}\prod_{k|n}(\eta(\tau/k))^{g_k} \\
=& D^{-1/2}(-i\tau)^{d/2}q^{-E_0}(1+O(q^{1/n}))
\end{aligned}
\tag{3}
$$

where

$$D = \prod_{k|n} k^{g_k} \tag{4a}$$

$$d = \sum_{k|n} g_k \tag{4b}$$

$$E_0 = -\frac{1}{24}\sum_{k|n}\frac{g_k}{k} \tag{4c}$$

where d is the number of unit eigenvalues of (1a).

A generalised Conway-Norton relationship.

Let us consider the conjugacy classes of .0 that satisfy the following two conditions

$$d = 0 \tag{5a}$$

$$E_0 > 0 \tag{5b}$$

The first condition implies that $\bar{g}$ acts on Λ without any non-trivial fixed vectors. The implications of the second condition are discussed below. The origin of both conditions is briefly discussed in the final remarks. A list of the 51 conjugacy classes obeying (5) is given in Table 1. The first column gives the standard Atlas [12] labelling of classes in $.0 =< \pm 1 > (.1)$ where .1 is the Conway simple group. The second column is the associated Frame shape. Let $h = \text{g.c.d.}\{k_1, k_2, ...\}$ for $k_i|n$ where $g_{k_i} \neq 0$. Then because $d = 0$, $\eta_{\bar{g}}(\tau)$ is a modular function (from (3)) invariant up to h roots of unity under a modular group Γ_g. This group is provided in column three employing the notation of ref.[1] where $\Gamma_g = \Gamma_0(n|h) + e_1, e_2, ...$ which is abbreviated to $n|h + e_1, e_2, ...$. Notice that $h|n$ and from (1b) that $h|24$ as observed in ref.[1]. In general, $\eta_{\bar{g}}(\tau)$ is fixed by a subgroup Γ'_g of index h in Γ_g which is of genus zero i.e. the compactification of the fundamental region H/Γ'_g is the Riemann sphere [1]. The corresponding hauptmodul, the unique (up to a constant) modular invariant meromorphic function with a simple pole (conventionally chosen at $q = 0$) is then just $1/\eta_{\bar{g}}(\tau)$.

The second condition (5b) above ensures that by specifying Γ_g, a unique conjugacy class of $.0$ is always obtained where the corresponding eta function $\eta_{\bar{g}}$ is inverted under the Fricke involution $\tau \to -1/nh\tau$ according to $\eta_{\bar{g}}(\tau) \to D^{-1/2}/\eta_{\bar{g}}(\tau)$ using (3). Each modular group Γ_g is also associated with a unique conjugacy class g of the Fischer-Griess Monster group M [3]. This is given in column four of Table 1 where Γ_g is the modular invariance group (up to h roots of unity) of the Thompson series for g [4,1]

$$
\begin{aligned}
T_g(\tau) =& \mathrm{Tr}_{V^\natural}(g q^{L_0}) \\
=& \frac{1}{q} + 0 + ...
\end{aligned}
\tag{6}
$$

where $V^\natural$ denotes the Moonshine Module of Frenkel, Lepowsky and Meurman (FLM) [5,6] and where L_0 is the grading operator (Virasoro Hamiltonian) where the elements of $V^\natural$ have integer grading -1,1,2,3,... (excluding 0). The automorphism group of $V^\natural$ which preserves this grading is M [5]. We also note that column four of Table 1 contains all the classes of M with modular group $\Gamma_g = n|h + e_1, e_2, ...$ with $e_i \neq n/h$ i.e. $T_g(\tau)$ is not invariant under the Fricke involution $\tau \to -1/nh\tau$. According to the recently proved [13] Moonshine conjectures [1] $T_g(\tau)$ is a hauptmodul for Γ'_g and thus

$$
T_g(\tau) = \frac{1}{\eta_{\bar{g}}(\tau)} - g_1 \tag{7}
$$

in each case given in Table 1 using the q expansions of (2) and (6).

We next conjecture a general relationship between the centraliser of $g \in M$, $C(g|M)$ and the centraliser of $\bar{g} \in .0$, $C(\bar{g}|.0)$. Let $G_n = C(\bar{g}|.0)/n$ i.e. $C(\bar{g}|.0) = n.G_n$ where n denotes the cyclic group generated by $\bar{g}$ and $A.B$ denotes a group with normal subgroup A and quotient group $B = A.B/A$. In columns six and seven of Table 1 the groups G_n and $C(g|M)$ are reproduced from refs.[14] and [1] respectively. In order to relate these centralisers, let us define a central extension $\hat{\Lambda}$ of Λ by the cyclic group $< \omega >$ (generated by $\omega = e^{2\pi i/n}$) for each $\bar{g}$ in Table 1 with exact sequence [15,16,6]

$$
1 \to < \omega > \to \hat{\Lambda} \to \Lambda \to 1 \tag{8}
$$

where the commutator of two elements $a, b \in \hat{\Lambda}$ lifted from $\alpha, \beta \in \Lambda$ is defined by

$$aba^{-1}b^{-1} = S(\alpha,\beta) \in < \omega > \tag{9}$$

with bilinear commutator map S given by

$$S(\alpha,\beta) = S(\beta,\alpha)^{-1} = \exp\,(2\pi i < \alpha, (1-\overline{g})^{-1}\beta >) \tag{10}$$

Since Λ is an integral lattice, this commutator also defines a central extension by $< \omega >$ of $L_{\overline{g}} = \Lambda/(1-\overline{g})\Lambda$ a finite abelian group of order $D = \det\,(1-\overline{g})$ (from (4a)). We denote this central extension by $\hat{L}_{\overline{g}}$. It is easy to see from the self-duality of Λ that Cent $(L_{\overline{g}}) =< \omega > \supseteq [L_{\overline{g}}, L_{\overline{g}}]$, the commutator subgroup [15]. In column five of Table 1 we display $\hat{L}_{\overline{g}}$ for each of the 51 automorphisms obeying (5).

If we now inspect columns five, six and seven of Table 1 we are led to the following conjecture:

Conjecture. *Let $\overline{g} \in .0$ satisfy (5) and let $\Gamma_g = n|h + e_1, e_2, \ldots$ ($e_i \neq n/h$) be the invariance modular group of the associated eta function $\eta_{\overline{g}}$. Let $g \in M$ have Thompson series with invariance modular group Γ_g also. Then $G_n = C(\overline{g}|.0)/n$ and $C(g|M)$ are related by*

$$C(g|M) = \hat{L}_{\overline{g}}.G_n \tag{11}$$

We make the following remarks concerning this conjecture.

(a) The extension of G_n by $\hat{L}_{\overline{g}}$ is consistent with a natural action of G_n on $\hat{L}_{\overline{g}}$ as follows. Since Λ is self-dual, the centre of $\hat{\Lambda}$ is given by all liftings of $(1-\overline{g})\Lambda$ using (10). We can therefore choose a section of $\hat{\Lambda}$, $\{c(\alpha)\}$, such that the abelian set $K = \{c((1-\overline{g})\alpha)\}$ closes to form a group so that $K \equiv (1-\overline{g})\Lambda$ and $\hat{\Lambda}/K \equiv \hat{L}_{\overline{g}}$. The group .0 of automorphisms of Λ is centrally extended to a group of automorphisms of $\hat{\Lambda}$ [15, 6]. In particular, $K \subset \hat{\Lambda}$ is invariant under a central extension of G_n (by $L_{\overline{g}}$) [9]. Thus G_n acts as a natural automorphism group on the quotient $\hat{L}_{\overline{g}} = \hat{\Lambda}/K$.

(b) For $\overline{g}$ of prime order p, (1b) and (5a) imply that $24 = (p-1)2r$ for $p = 2,3,5,7,13$ and $r = 12,6,3,2,1$ with $g_p = -g_1 = 2r$. Then $L_{\overline{g}} = p^{2r}$ is elementary abelian and $\hat{L}_{\overline{g}} = p^{1+2r}$ is extra-special [1,6]. In these cases, as originally observed by Conway and Norton [1], (11) is clearly true from Table 1.

(c) For 11 non-prime cases, (11) can also be verified directly from Table 1. However, G_n and $C(g|M)$ are not explicitly available in refs.[14] and [1] for all $\overline{g}$ and g listed. Nevertheless, the order of these groups, as displayed in Table 1, is certainly consistent with (11) which strongly supports the conjecture in general.

(d) The conjecture (11) has an interpretation in string theory to which it owes its origin. The details of this will appear elsewhere [17] where we argue that each of the 51 automorphisms of Table 1 can be employed to construct the FLM Moonshine Module $V^\natural$. In the original work of FLM [5,6], $V^\natural$ is constructed as an orbifold [7,8] conformal field theory based on the lattice reflection automorphism $\overline{r}$ of a closed bosonic string propagating on the 24 dimensional torus R^{24}/Λ [18]. $\overline{r}$ is the first automorphism appearing in Table 1. In that construction, there is a natural order two automorphism which belongs to the Monster class $2-$. The centraliser $C(2-|M)$ can then be found explicitly to be exactly $2^{1+24}.(.1)$ as given by (11). FLM have also conjectured [6] that $V^\natural$, together with its associated vertex operator algebra, is the unique modular invariant bosonic conformal field theory with partition function $T_1(\tau) = J(\tau)$ [11], the hauptmodul for the modular group SL$(2, Z)$ with a simple pole at $q = 0$ and zero constant term. Each of the other 50 lattice automorphisms of Table 1 can also be used in orbifold constructions to obtain a conformal field theory with this partition function. For the 37 classes with associated modular group $\Gamma_g = n + e_1, e_2, \ldots$ $(h = 1)$ this is performed in the standard way [9]. The remaining cases require a Gepner-like construction [19] to obtain a modular invariant orbifold theory [17]. The absence of any L_0 level zero states in these constructions is guaranteed by the constraints (5) [9]. There is also a natural order n automorphism g of the orbifold conformal field theory for each $\bar{g}$ of Table 1 where the Thompson series for g can be shown to be given by (7) (see ref. [9] for the 37 cases with $h = 1$). The centraliser $C(g|M)$ can then be shown to be precisely $\hat{L}_{\bar{g}}.G_n$ in agreement with (11). Therefore the conjectured centraliser relationship (11) can be understood in general, given that the orbifold constructions based on the automorphisms of Table 1 do indeed reproduce $V^\natural$. On the other hand, given that (11) is true, we have strong evidence to support the FLM uniqueness conjecture.

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$\bar{g} \in .O$	$\bar{g}$ Frame Shape	Γ_g	$g \in M$	$\hat{L}_{\bar{g}}$	G_n	$C(g\|M)$
$-1A$	$2^{24}/1^{24}$	$2-$	$2B$	2^{1+24}	$.1$	$2^{1+24}(.1)$
$3A$	$3^{12}/1^{12}$	$3-$	$3B$	3^{1+12}	2.Sz	3^{1+12}.2.Sz
$4A$	$4^8/1^8$	$4-$	$4C$	4.4^8	$2.2^6.S_6(2)$	$4.2^{15}.2^8.S_6(2)$
$\pm 2B$	$4^{12}/2^{12}$	$4\|2-$	$4D$	4.2^{12}	$G_2(4).2$	$4.2^{12}.G_2(4).2$
$5A$	$5^6/1^6$	$5-$	$5B$	5^{1+6}	2.HJ	5^{1+6}.2.HJ
$-3B$	$2^6 6^6/1^6 3^6$	$6+3$	$6C$	$2^{1+12} \times 3$	$3.U_4(3).2$	$2^{1+12}.3^2.U_4(3).2$
$6A$	$3^4 6^4/1^4 2^4$	$6+2$	$6D$	$2 \times 3^{1+8}$	$2^{1+6}.U_4(2)$	$2.3^{1+8}.2^{1+6}.U_4(2)$
$6D$	$2.6^5/1^5 3$	$6-$	$6E$	$2^{1+6} \times 3^{1+4}$	$2.U_4(2)$	$2.3^{1+4}.2^{1+6}.U_4(2)$
$-3D$	$6^8/3^8$	$6\|3-$	$6F$	$3 \times 2^{1+8}$	A_9	$3 \times 2^{1+8}.A_9$
$7A$	$7^4/1^4$	$7-$	$7B$	7^{1+4}	$2.A_7$	$7^{1+4}.2.A_7$
± 8	$8^4/2^4$	$8\|2-$	$8D$	8.4^4	$2.2^4.A_6$	$8.2^9.2^4.A_6$
$8C$	$2^2 8^4/1^4 4^2$	$8-$	$8E$	$8.(8^2 \times 4^2)$	$[2^9.3]$	$[2^{22}.3]$
$\pm 4E$	$8^6/4^6$	$8\|4-$	$8F$	8.2^6	$U_3(3)$	$8.2^6.U_3(3)$
$9A$	$9^3/1^3$	$9-$	$9B$	$9.(9^2 \times 3^2)$	$[2^4.3^3]$	$[2^4.3^{11}]$
$-5B$	$2^4 10^4/1^4 5^4$	$10+5$	$10B$	$5 \times 2^{1+8}$	$(A_5 \times A_5).2$	$5 \times 2^{1+8}.(A_5 \times A_5).2$
$10A$	$5^2 10^2/1^2 2^2$	$10+2$	$10C$	$2 \times 5^{1+4}$	$2^{1+4}.A_5$	$2.5^{1+4}.2^{1+4}.A_5$
$10E$	$2.10^3/1^3 5$	$10-$	$10E$	$2^{1+4} \times 5^{1+2}$	$2A_5$	$2.5^{1+2}.2^{1+4}.A_5$
$-12A$	$2^4 3^4 12^4/1^4 4^4 6^4$	$12+4$	$12B$	$4 \times 3^{1+4}$	$2.2^4.S_6$	$[2^{11}.3^7.5]$
$12E$	$4^2 12^2/1^2 3^2$	$12+3$	$12E$	$4.4^4 \times 3$	$[2^5.3^2]$	$[2^{15}.3^3]$
$\pm 12C$	$6^2 12^2/2^2 4^2$	$12\|2+2$	$12G$	$4 \times 3^{1+4}$	$[2^7.3]$	$[2^9.3^6]$
$12K$	$2^2 3.12^3/1^3 4.6^2$	$12-$	$12I$	$4.4^2 \times 3^{1+2}$	$[2^4.3]$	$[2^{10}.3^4]$
$\pm 6H$	$12^4/6^4$	$12\|6-$	$12J$	3×4.2^4	$A_5 \times 2$	$[2^9.3^2 5]$
$13A$	$13^2/1^2$	$13-$	$13B$	13^{1+2}	$2.A_4$	$13^{1+2}.2.A_4$
$-7B$	$2^3 14^3/1^3 7^3$	$14+7$	$14B$	$7 \times 2^{1+6}$	$L_2(7)$	$[2^{10}.3.7^2]$
$15B$	$3^2 15^2/1^2 5^2$	$15+5$	$15B$	$5 \times 3^{1+4}$	$2.A_5$	$[2^3.3^6.5^2]$
$15C$	$15^2/3^2$	$15\|3-$	$15D$	$3 \times 5^{1+2}$	$2.A_4$	$[2^3.3^2.5^3]$
$16B$	$2.16^2/1^2 8$	$16-$	$16B$	16.8^2	$[2^3]$	$[2^{13}]$
$18A$	$9.18/1.2$	$18+2$	$18A$	2×9.9^2	$[2^3.3]$	$[2^4.3^7]$
$-9C$	$2^3 3^2 18^3/1^3 6^2 9^3$	$18+9$	$18C$	$2^{1+4} \times 9$	$[2.3^3]$	$[2^6.3^5]$
$18B$	$2.3.18^2/1^2 6.9$	$18-$	$18D$	$2^{1+2} \times 9.3^2$	2.3	$[2^4.3^5]$
$-20A$	$2^2 5^2 20^2/1^2 4^2 10^2$	$20+4$	$20C$	$4 \times 5^{1+2}$	$2.S_4$	$[2^6.3.5^3]$
$\pm 10C$	$4^2 20^2/2^2 10^2$	$20\|2+5$	$20D$	$4.2^4 \times 5$	A_5	$[2^8.3.5^2]$

$\bar{g} \in .0$	$\bar{g}$ Frame Shape	Γ_g	$g \in M$	$\hat{L}_{\bar{g}}$	G_n	$C(g\|M)$
$21B$	$7.21/1.3$	$21+3$	$21B$	$3 \times 7^{1+2}$	2.3	$[2.3^2.7^3]$
$-11A$	$2^2 22^2/1^2 11^2$	$22+11$	$22B$	$2^{1+4} \times 11$	S_3	$[2^6.3.11]$
$-24B$	$2.3^2 4.24^2/1^2 6.8^2 12$	$24+8$	$24C$	$8 \times 3^{1+2}$	$[2^4]$	$[2^7.3^3]$
$\pm 24C$	$8.24/2.6$	$24\|2+3$	$24D$	3×8.4^2	$[2.3]$	$[2^8.3^2]$
$\pm 24D$	$12.24/4.8$	$24\|4+2$	$24G$	$8 \times 3^{1+2}$	$[2^2]$	$[2^5.3^3]$
$\pm 12L$	$24^2/12^2$	$24\|12-$	$24J$	3×8.2^2	3	$[2^5.3^2]$
$28A$	$4.28/1.7$	$28+7$	$28C$	$4.4^2 \times 7$	2	$[2^7.7]$
$-15A$	$2^3 3^3 5^3 30^3/1^3 6^3 10^3 15^3$	$30+6,10,15$	$30A$	$2 \times 3 \times 5$	A_6	$[2^4.3^3.5^2]$
$-15D$	$2.6.10.30/1.3.5.15$	$30+3,5,15$	$30C$	$2^{1+4} \times 3 \times 5$	S_3	$[2^6.3^2.5]$
$-15E$	$2^2 3.5.30^2/1^2 6.10.15^2$	$30+15$	$30G$	$2^{1+2} \times 3 \times 5$	2	$[2^4.3.5]$
$33A$	$3.33/1.11$	$33+11$	$33A$	$3^{1+2} \times 11$	2	$[2.3^3.11]$
$-36A$	$2.9.36/1.4.18$	$36+4$	$36B$	4×9.3^2	2	$[2^3.3^4]$
$-21A$	$2^2 3^2 7^2 42^2/1^2 6^2 14^2 21^2$	$42+6,14,21$	$42B$	$2 \times 3 \times 7$	A_4	$[2^3.3^2.7]$
$-21C$	$6.42/3.21$	$42\|3+7$	$42C$	$2^{1+2} \times 3 \times 7$	1	$[2^3.3.7]$
$-23A$	$2.46/1.23$	$46+23$	$46AB$	$2^{1+2} \times 23$	1	$[2^3.23]$
$60A$	$3.4.5.60/1.12.15.20$	$60+12,15,20$	$60D$	$4 \times 3 \times 5$	2	$[2^3.3.5]$
$-35A$	$2.5.7.70/1.10.14.35$	$70+10,14,35$	$70B$	$2 \times 5 \times 7$	1	$[2.5.7]$
$-39A$	$2.3.13.78/1.6.26.39$	$78+6,26,39$	$78B$	$2 \times 3 \times 13$	1	$[2.3.13]$
$\pm 42A$	$4.6.14.84/2.12.28.42$	$84\|2+6,14,21$	$84B$	$4 \times 3 \times 7$	1	$[2^2.3.7]$

Table 1

The 51 Conway group classes that obey (5) appear in columns 1 and 2 in Atlas and Frame shape notation respectively. The Monster group class in column 4 has the same associated modular group Γ_g given in column 3. The groups appearing in columns 5, 6 and 7 are expressed in terms of standard Atlas finite groups where n denotes a cyclic group of that order and p^{1+2d} denotes an extra-special group. The groups denoted by $[p_1^a.p_2^b...]$ are unknown with the order as given.

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